

# The property of maximal transcendentality: calculation of master integrals

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## Abstract

We review the results having the property of maximal transcendentality.

## 1 Introduction

Recently discovered that a popular property of maximal transcendentality, which was introduced in [1] for the Balitsky-Fadin-Kuraev-Lipatov (BFKL) kernel [2, 3] in the  $\mathcal{N} = 4$  Supersymmetric Yang-Mills (SYM) model [4], is also applicable to the amplitudes, form-factors and correlation functions (see [5, 6, 7] and discussions and references therein).

The aim of this short paper is to show this property in the results for the anomalous dimension (AD) matrix of the twist-2 Wilson operators and to demonstrate a similar feature in the results for so-called master integrals [8].

## 2 ADs in $\mathcal{N} = 4$ SYM

The ADs govern the Bjorken scaling violation for parton distributions ( $\equiv$  matrix elements of the twist-2 Wilson operators) in a framework of Quantum Chromodynamics (QCD).

The BFKL and Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [9] DGLAP equations resum, respectively, the most important contributions  $\sim \alpha_s \ln(1/x)$  and  $\sim \alpha_s \ln(Q^2/\Lambda^2)$  in different kinematical regions of the Bjorken variable  $x$  and the “mass”  $Q^2$  of the virtual photon in the deep inelastic lepton-hadron scattering (see Fig. 1 for the muon-nucleon case) and, thus, they are the cornerstone in analyses of the experimental data from lepton-nucleon and nucleon-nucleon scattering processes. In the supersymmetric generalization of QCD the equations are simplified drastically (see [10]).

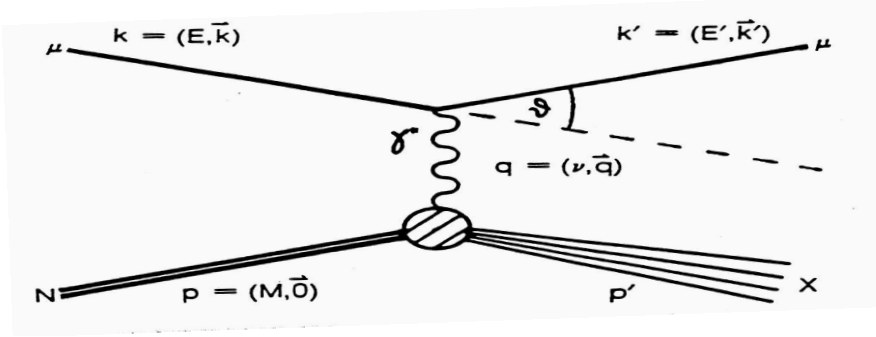


Figure 1: The deep inelastic muon-nucleon scattering, where  $k$ ,  $q$  and  $p$  are the muon, photon and nucleon momenta, respectively. In the deep inelastic kinematics,  $p^2 = M^2 \rightarrow 0$ , where  $M$  is the nucleon mass. The standard variables are  $Q^2 = -q^2 > 0$  and the Bjorken variable  $x = Q^2/(2pq)$ , where  $Q^2$  is the “mass” of the virtual photon and  $x$  is the part of the nucleon momentum carried by the colliding parton (quark or gluon).

## 2.1 Leading order

The elements of the leading order (LO) AD matrix in the  $\mathcal{N} = 4$  SYM have the following form (see [11]):

$$\begin{aligned}
\gamma_{gg}^{(0)}(j) &= 4 \left( \Psi(1) - \Psi(j-1) - \frac{2}{j} + \frac{1}{j+1} - \frac{1}{j+2} \right), \\
\gamma_{\lambda g}^{(0)}(j) &= 8 \left( \frac{1}{j} - \frac{2}{j+1} + \frac{2}{j+2} \right), & \gamma_{\varphi g}^{(0)}(j) &= 12 \left( \frac{1}{j+1} - \frac{1}{j+2} \right), \\
\gamma_{g\lambda}^{(0)}(j) &= 2 \left( \frac{2}{j-1} - \frac{2}{j} + \frac{1}{j+1} \right), & \gamma_{q\varphi}^{(0)}(j) &= \frac{8}{j}, \\
\gamma_{\lambda\lambda}^{(0)}(j) &= 4 \left( \Psi(1) - \Psi(j) + \frac{1}{j} - \frac{2}{j+1} \right), & \gamma_{\varphi\lambda}^{(0)}(j) &= \frac{6}{j+1}, \\
\gamma_{\varphi\varphi}^{(0)}(j) &= 4(\Psi(1) - \Psi(j+1)), & \gamma_{g\varphi}^{(0)}(j) &= 4 \left( \frac{1}{j-1} - \frac{1}{j} \right), \quad (1)
\end{aligned}$$

where  $j$  is the Mellin moment (or spin) number.

The matrix, based on the ADs (1), can be diagonalized [11, 1]:

$$\left[ D\Gamma D^{-1} \right]_{\text{unpol}}^{N=4} = \begin{vmatrix} -4S_1(j-2) & 0 & 0 \\ 0 & -4S_1(j) & 0 \\ 0 & 0 & -4S_1(j+2) \end{vmatrix},$$

where  $S_1(j)$  is defined below in (2).

Thus, the LO ADs of all multiplicatively renormalized Wilson operators can be extracted through one universal function

$$\gamma_{uni}^{(0)}(j) = -4S(j-2) \equiv -4\left(\Psi(j-1) - \Psi(1)\right) \equiv -4\sum_{r=1}^{j-2} \frac{1}{r}.$$

Same results can be obtained also for spin-dependent case (see [11, 1]).

## 2.2 Method to get the universal AD

Let us to introduce the transcendentality level  $i$  for the harmonic sums

$$S_{\pm a}(j) = \sum_{m=1}^j \frac{(\pm 1)^m}{m^a}, \quad S_{\pm a, \pm b, \pm c, \dots}(j) = \sum_{m=1}^j \frac{(\pm 1)^m}{m^a} S_{\pm b, \pm c, \dots}(m), \quad (2)$$

and Euler-Zagier constants

$$\zeta(\pm a) = \sum_{m=1}^{\infty} \frac{(\pm 1)^m}{m^a}, \quad \zeta(\pm a, \pm b, \pm c, \dots) = \sum_{m=1}^{\infty} \frac{(\pm 1)^m}{m^a} S_{\pm b, \pm c, \dots}(m-1), \quad (3)$$

in the following way

$$S_{\pm a, \pm b, \pm c, \dots}(j) \sim \zeta(\pm a, \pm b, \pm c, \dots) \sim 1/j^i, \quad (i = a + b + c + \dots) \quad (4)$$

Then, the basic functions  $\gamma_{uni}^{(0)}(j)$ ,  $\gamma_{uni}^{(1)}(j)$  and  $\gamma_{uni}^{(2)}(j)$  are assumed to be of the types  $\sim 1/j^i$  with the levels  $i = 1$ ,  $i = 3$  and  $i = 5$ , respectively. An exception could be for the terms appearing at a given order from previous orders of the perturbation theory. Such contributions could be generated and/or removed by an approximate finite renormalization of the coupling constant. But these terms do not appear in the  $\overline{\text{DR}}$ -scheme [12].

It is known, that at the LO, the next-to-leading order (NLO) and the next-to-next-to-leading order (NNLO) approximations (with the SUSY relation for the QCD color factors  $C_F = C_A = N_c$ ) the most complicated contributions (with  $i = 1$ , 3 and 5, respectively) are the same for all LO, NLO and NNLO ADs in QCD [13] and for the LO and NLO scalar-scalar ADs [14]. This property allows one to find the universal ADs  $\gamma_{uni}^{(0)}(j)$ ,  $\gamma_{uni}^{(1)}(j)$  and  $\gamma_{uni}^{(2)}(j)$  without knowing all elements of the AD matrix [1], which was verified for  $\gamma_{uni}^{(1)}(j)$  by the exact calculations in [14].

## 2.3 Universal AD for $\mathcal{N} = 4$ SYM

The final three-loop result <sup>1</sup> for the universal AD  $\gamma_{uni}(j)$  for  $\mathcal{N} = 4$  SYM is [15]

$$\gamma(j) \equiv \gamma_{uni}(j) = \hat{a} \gamma_{uni}^{(0)}(j) + \hat{a}^2 \gamma_{uni}^{(1)}(j) + \hat{a}^3 \gamma_{uni}^{(2)}(j) + \dots, \quad \hat{a} = \frac{\alpha N_c}{4\pi}, \quad (5)$$

where

$$\frac{1}{4} \gamma_{uni}^{(0)}(j+2) = -S_1, \quad (6)$$

$$\frac{1}{8} \gamma_{uni}^{(1)}(j+2) = \left( S_3 + \bar{S}_{-3} \right) - 2 \bar{S}_{-2,1} + 2 S_1 \left( S_2 + \bar{S}_{-2} \right), \quad (7)$$

$$\begin{aligned} \frac{1}{32} \gamma_{uni}^{(2)}(j+2) = & 2 \bar{S}_{-3} S_2 - S_5 - 2 \bar{S}_{-2} S_3 - 3 \bar{S}_{-5} + 24 \bar{S}_{-2,1,1,1} \\ & + 6 \left( \bar{S}_{-4,1} + \bar{S}_{-3,2} + \bar{S}_{-2,3} \right) - 12 \left( \bar{S}_{-3,1,1} + \bar{S}_{-2,1,2} + \bar{S}_{-2,2,1} \right) \\ & - \left( S_2 + 2 S_1^2 \right) \left( 3 \bar{S}_{-3} + S_3 - 2 \bar{S}_{-2,1} \right) - S_1 \left( 8 \bar{S}_{-4} + \bar{S}_{-2}^2 \right. \\ & \left. + 4 S_2 \bar{S}_{-2} + 2 S_2^2 + 3 S_4 - 12 \bar{S}_{-3,1} - 10 \bar{S}_{-2,2} + 16 \bar{S}_{-2,1,1} \right) \end{aligned} \quad (8)$$

with  $S_{\pm a, \pm b, \pm c, \dots} \equiv S_{\pm a, \pm b, \pm c, \dots}(j)$  and

$$\bar{S}_{-a, b, c, \dots}(j) = (-1)^j S_{-a, b, c, \dots}(j) + S_{-a, b, c, \dots}(\infty) \left( 1 - (-1)^j \right). \quad (9)$$

The expression (9) is the analytical continuation (to real and complex  $j$ ) [16] of the harmonic sums  $S_{-a, b, c, \dots}(j)$ .

The results for  $\gamma_{uni}^{(3)}(j)$  [17, 18] and  $\gamma_{uni}^{(4)}(j)$  [19] can be obtained from the long-range asymptotic Bethe equations [20] for twist-two operators and the additional contribution of the wrapping corrections.

## 3 Calculation of Feynman integrals

The arguments similar to ones considered in [1], give a possibility to calculate a large class of Feynman diagrams, so-called master-integrals [8]. Let us consider it in some details.

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<sup>1</sup> Note, that in accordance with Ref. [3] our normalization of  $\gamma(j)$  contains the extra factor  $-1/2$  in comparison with the standard normalization (see [1]) and differs by sign in comparison with one from Ref. [13].

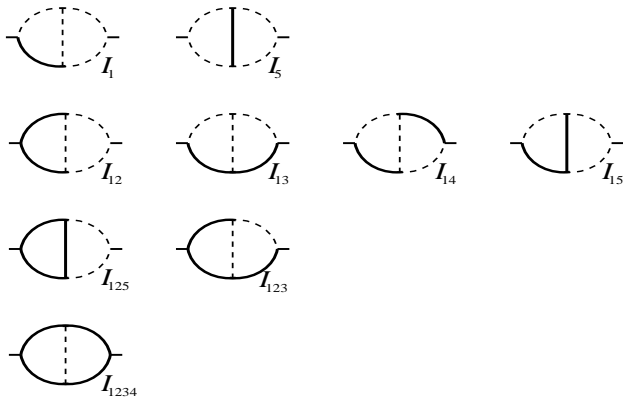


Fig. 2

Application of the integration-by-part (IBP) procedure [22] to loop internal momenta leads to relations between different Feynman integrals (FIs) and, thus, to necessity to calculate only some of them, which in a sense, are independent (see [23]). These independent diagrams (which were chosen quite arbitrary, of course) are called the master-integrals [8].

The application of the IBP procedure [22] to the master-integrals themselves leads to the differential equations [23, 24] for them with the inhomogeneous terms (ITs) containing less complicated diagrams.<sup>2</sup> The application of the IBP procedure to the diagrams for ITs leads to the new differential equations for them with the new ITs containing even farther less complicated diagrams. Repeating the procedure several times, at a last step one can obtain the ITs containing mostly tadpoles which can be calculated in-turn very easily.

Solving the differential equations at this last step, one can reproduce the diagrams for ITs of the differential equations at the previous step. Repeating the procedure several times one can obtain the results for the initial FIs.

This scheme has been used successfully for calculation of two-loop two-point [23, 24, 26] and three-point diagrams [27, 21] with one nonzero mass. This procedure is very powerful but quite complicated. There are, however, some simplifications, which are based on the series representations of FIs.

Indeed, the inverse-mass expansion of two-loop two-point (see Fig. 2)

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<sup>2</sup>The “less complicated diagrams” contain usually less number of propagators and sometimes they can be represented as diagrams with less number of loops and with some “effective masses” (see, for example, [25] and references therein).

and three-point diagrams (see Fig. 3)<sup>3</sup> with one nonzero mass (massless and massive propagators are shown as dashed and solid lines, respectively), can be considered as

$$\text{FI} = \frac{\hat{N}}{q^{2\alpha}} \sum_{n=1} C_n \frac{(\eta x)^n}{n^c} \left\{ F_0(n) + \left[ \ln(-x) F_{1,1}(n) + \frac{1}{\varepsilon} F_{1,2}(n) \right] \right. \quad (10)$$

$$\left. + \left[ \ln^2(-x) F_{2,1}(n) + \frac{1}{\varepsilon} \ln(-x) F_{2,2}(n) + \frac{1}{\varepsilon^2} F_{2,3}(n) + \zeta(2) F_{2,4}(n) \right] + \cdots \right\},$$

where  $x = q^2/m^2$ ,  $\eta = 1$  or  $-1$ ,  $c = 0, 1$  and  $2$ , and  $\alpha = 1$  and  $2$  for two-point and three-point cases, respectively.

Here the normalization  $\hat{N} = (\bar{\mu}^2/m^2)^{2\varepsilon}$ , where  $\bar{\mu} = 4\pi e^{-\gamma_E} \mu$  is in the standard  $\overline{MS}$ -scheme and  $\gamma_E$  is the Euler constant. Moreover, the space-time dimension is  $D = 4 - 2\varepsilon$  and

$$C_n = 1, \quad \text{and} \quad C_n = \frac{(n!)^2}{(2n)!} \equiv \hat{C}_n \quad (11)$$

for diagrams with two-massive-particle-cuts ( $2m$ -cuts). For the diagrams with one-massive-particle-cuts ( $m$ -cuts)  $C_n = 1$ .

For  $m$ -cut case, the coefficients  $F_{N,k}(n)$  should have the form

$$F_{N,k}(n) \sim \frac{S_{\pm a, \dots}}{n^b}, \quad (12)$$

where  $S_{\pm a, \dots} \equiv S_{\pm a, \dots}(j-1)$  are harmonic sums in (2).

For  $2m$ -cut case, the coefficients  $F_{N,k}(n)$  should have the form<sup>4</sup>

$$F_{N,k}(n) \sim \frac{S_{\pm a, \dots}}{n^b}, \frac{V_{a, \dots}}{n^b}, \frac{W_{a, \dots}}{n^b}, \quad (13)$$

where  $V_{\pm a, \dots} \equiv V_{\pm a, \dots}(j-1)$  and  $W_{\pm a, \dots} \equiv W_{\pm a, \dots}(j-1)$  with

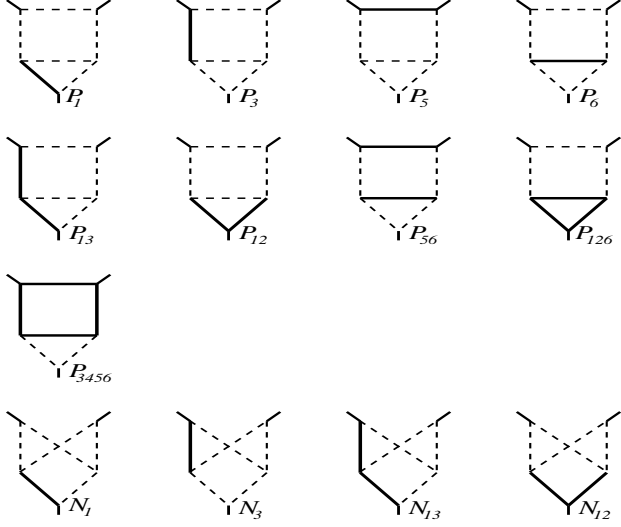
$$V_a(j) = \sum_{m=1}^j \frac{\hat{C}_m}{m^a}, \quad V_{a,b,c,\dots}(j) = \sum_{m=1}^j \frac{\hat{C}_m}{m^a} S_{b,c,\dots}(m), \quad (14)$$

$$W_a(j) = \sum_{m=1}^j \frac{\hat{C}_m^{-1}}{m^a}, \quad W_{a,b,c,\dots}(j) = \sum_{m=1}^j \frac{\hat{C}_m^{-1}}{m^a} S_{b,c,\dots}(m), \quad (15)$$

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<sup>3</sup>We consider only three-point diagrams with independent upward momenta  $q_1$  and  $q_2$ , which obey the conditions  $q_1^2 = q_2^2 = 0$  and  $(q_1 + q_2)^2 \equiv q^2 \neq 0$ , where  $q$  is downward momentum.

<sup>4</sup>Really, there are even more complicated terms as ones in Eqs. (58) and (59) of [21], which come from other  $\eta$  values in (10). However, they are outside of our present consideration.



**Fig. 3**

The terms  $\sim V_{a,\dots}$  and  $\sim W_{a,\dots}$  can come only in the  $2m$ -cut case. The origin of the appearance of these terms is the product of series (10) with the different coefficients  $C_n = 1$  and  $C_n = \hat{C}_n$ .

As an example, consider two-loop two-point diagrams  $I_1$  and  $I_{12}$  shown in Fig. 2 and studied in [21]

$$I_1 = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{x^n}{n} \left\{ \frac{1}{2} \ln^2(-x) - \frac{2}{n} \ln(-x) + \zeta(2) + 2S_2 - 2\frac{S_1}{n} + \frac{3}{n^2} \right\} \quad (16)$$

$$I_{12} = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{x^n}{n^2} \left\{ \frac{1}{n} + \frac{(n!)^2}{(2n)!} \left( -2 \ln(-x) - 3W_1 + \frac{2}{n} \right) \right\}. \quad (17)$$

From (16) one can see that the corresponding functions  $F_{N,k}(n)$  have the form

$$F_{N,k}(n) \sim \frac{1}{n^{2-N}}, \quad (N \geq 2), \quad (18)$$

if we introduce the following complexity of the sums ( $\Phi = (S, V, W)$ )

$$\Phi_{\pm a} \sim \Phi_{\pm a_1, \pm a_2} \sim \Phi_{\pm a_1, \pm a_2, \dots, \pm a_m} \sim \zeta_a \sim \frac{1}{n^a}, \quad \left( \sum_{i=1}^m a_i = a \right). \quad (19)$$

In Eq. (17),

$$F_{N,k}(n) \sim \frac{1}{n^{1-N}}, \quad (N \geq 1), \quad (20)$$

since now the factor  $1/n^2$  has been already extracted.

So, Eqs. (16)-(17) show that the functions  $F_{N,k}(n)$  should have the following form

$$\frac{1}{n^c} F_{N,k}(n) \sim \frac{1}{n^{3-N}}, \quad (N \geq 2) \quad (21)$$

and the number  $3 - N$  defines the level of transcendentality (or complexity) of the coefficients  $F_{N,k}(n)$ . The property reduces strongly the number of the possible elements in  $F_{N,k}(n)$ . The level of transcendentality decreases if we consider the singular parts of diagrams and/or coefficients in front of  $\zeta$ -functions and of logarithm powers.

Other  $I$ -type integrals in [21] have similar form. They have been calculated exactly by differential equation method [23, 24].

Now we consider two-loop three-point diagrams,  $P_5$  and  $P_{12}$  shown in Fig. 3 and calculated in [21]:

$$\begin{aligned} P_5 &= \frac{\hat{N}}{(q^2)^2} \sum_{n=1} \frac{(-x)^n}{n} \left\{ -6\zeta_3 + 2(S_1\zeta_2 + 6S_3 - 2S_1S_2 + 4\frac{S_2}{n} - \frac{S_1^2}{n} + 2\frac{S_1}{n^2} \right. \\ &\quad \left. + \left( -4S_2 + S_1^2 - 2\frac{S_1}{n} \right) \ln(-x) + S_1 \ln^2(-x) \right\}, \quad (22) \\ P_{12} &= \frac{\hat{N}}{q^2} \sum_{n=1} \frac{x^n}{n^2} \frac{(n!)^2}{(2n)!} \left\{ \frac{2}{\varepsilon^2} + \frac{2}{\varepsilon} \left( S_1 - 3W_1 + \frac{1}{n} - \ln(-x) \right) + 12W_2 - 18W_{1,1} \right. \\ &\quad \left. - 13S_2 + S_1^2 - 6S_1W_1 + 2\frac{S_1}{n} + \frac{2}{n^2} - 2 \left( S_1 + \frac{1}{n} \right) \ln(-x) + \ln^2(-x) \right\}, \end{aligned}$$

Now the coefficients  $F_{N,k}(n)$  have the form

$$\frac{1}{n^c} F_{N,k}(n) \sim \frac{1}{n^{4-N}}, \quad (N \geq 3), \quad (23)$$

The diagram  $P_5$  (and also  $P_1$ ,  $P_3$  and  $P_6$  in [21]) have been calculated exactly by differential equation method [23, 24]. To find the results for  $P_{12}$  (and also all others in [21]) we have used the knowledge of the several  $n$  terms in the inverse-mass expansion (10) (usually less than  $n = 100$ ) and the following arguments:



- The coefficients should have the structure (23) with the rule (19). The condition (23) reduces strongly the number of possible harmonic sums. It should be related with the specific form of the differential equations for the considered master integrals, like

$$\left(\bar{k}\varepsilon + m^2 \frac{d}{dm^2}\right) \text{ FI} = \text{less complicated diagrams} ,$$

with some  $\bar{k}$  values. We note that for many other master integrals (for example, for sunsets with two massive lines in [23, 26, 25]) the property (23) is violated: the coefficients  $F_{N,k}(n)$  contain sums with different levels of complexity.<sup>5</sup>

- If a two-loop two-point diagram with the “similar topology” (for example,  $I_{12}$  for  $P_{12}$  and so on) has been already calculated, we should consider a similar set of basic elements for corresponding  $F_{N,k}(n)$  of two-loop three-point diagrams but with the higher level of complexity.
- Let the considered diagram contain singularities and/or powers of logarithms. Because in front of the leading singularity, or the largest power of logarithm, or the largest  $\zeta$ -function the coefficients are very simple, they can be often predicted directly from the first several terms of expansion.

Moreover, often we can calculate the singular part using another technique (see [21] for extraction of  $\sim W_1(n)$  part). Then we should expand the singular parts, find the basic elements and try to use them (with the corresponding increase of the level of complexity) to predict the regular part of the diagram. If we have to find the  $\varepsilon$ -suppressed terms, we should increase the level of complexity for the corresponding basic elements.

Later, using the ansatz for  $F_{N,k}(n)$  and several terms (usually, less than 100) in the above expression, which can be calculated exactly, we obtain the system of algebraic equations for the parameters of the ansatz. Solving the system, we can obtain the analytical results for FI without exact calculations.

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<sup>5</sup>Really, Refs. [23, 26] contain the Nilson polylogarithms, whose sum of indices relates directly to the level of transcendentality ( $4 - N$ ). The representation of the series (16)-(17) and (22), containing  $S_{\pm a, \dots}$ , as polylogarithms can be found in [21] for  $m$ -cut case and in [28] for  $2m$ -cut one, respectively.

To check the results, it is needed only to calculate a few more terms in the above inverse-mass expansion (10) and compare them with the predictions of our ansatz with the above fixed coefficients.

So, the considered arguments give a possibility to find the results for many complicated two-loop three-point diagrams without direct calculations. Some variations of the procedure have been successfully used for calculating the Feynman diagrams for many processes (see [27, 21, 25, 29]).

Note that the properties similar to (21) and (23) have been observed recently [7] in the so-called double operator-product-expansion limit of some four-point diagrams. These diagrams are encoded the quantum corrections to the four-point correlator and have been considered in [7] up-to three-loop level of accuracy.

## 4 Conclusion

In the first part of this short review we presented the universal AD  $\gamma_{uni}(j)$  for the  $\mathcal{N} = 4$  supersymmetric gauge theory up to the NNLO approximation. All the results have been obtained with using of the *transcendentality principle*. At the first three orders, the universal anomalous dimension have been extracted from the corresponding QCD calculations. The results for four and five loops have been obtained from the long-range asymptotic Bethe equations together with some additional terms, so-called *wrapping corrections*, coming in agreement with Luscher approach.

The second part contains the consideration of so-called master integrals, which obey also to the similar *transcendentality principle* (19). Its application leads to the possibility to get the results for most of master integrals without direct calculations.

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